Session 2: Introduction to Probability
Foundations of Quantitative Ecology (EEOB 8896.11)

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- Environments are constantly changing! Thus, *Environmental (extrinsic)* noise is also ubiquitous.
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In short, **ALL models of living systems are, or are simplifications of, stochastic models.**
So, *how/why* are probability concepts widely used in science?

A few examples...

- Simulation (sampling from distributions, e.g., to mimic data)
- Qualitative properties (expected values, expected deviations)
- Approximation (Law of Large Numbers)
- Deriving relationships (e.g., functional forms) and other models
- Statistics (e.g., Maximum Likelihood = Maximum density!)
Conceptual Framework

- DATA
- MODEL Single Distribution
- Statistics
- Distributions/Random Variables (Family of models)

Model Simulation
Example: Linear Regression $y = \beta_0 + \beta_1 x + \epsilon$, where $\epsilon \sim N(0, \sigma)$.

```r
set.seed(1492); ## ?set.seed or ask me :-)
b0=2; b1=1; sig=2; y=b0+b1*x+rnorm(length(x),0,sig);

## Error: object 'x' not found

plot(x,y,pch=19); abline(b0,b1);

## Error: object 'x' not found

abline(lm(y~x,data=data.frame(x,y)),lty=2)

## Error: object 'x' not found
```
Motivation

Distribution Properties

Mean vs Expected value? Standard Deviation? Moment Generating Function? **Conjugate Distributions** (Bayesian prior & posterior)?

```r
x = rbinom(100, 20, p = 0.2)
mean(x)  ## Compare mean(x) vs. E(x)=n*p
## [1] 4.04
sd(x)    ## Compare sd(x)^2 vs. Var(x)=n*p*(1-p)
## [1] 1.693
sqrt(20 * 0.2 * (1 - 0.2))
## [1] 1.789
```

General mathematical results (aka Analytical results) are really powerful, *if* we can find them! They give general answers to our scientific questions, guide biological intuition, and speed up computations.
Approximation & Deriving Other Models

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**General Results:** Model approximation (or considering special cases of a model) can yield well understood (approximate) models for which useful, general results already exist!
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To see how all this works, we need to start looking at probability distributions in detail.
Conceptual Framework

DATA

Model Simulation

MODEL
Single Distribution

Statistics

Distributions/
Random Variables
(Family of models)
Motivation

Probability Concepts

Distribution & Density Functions

Two ways to think about the Normal distribution (a continuous distribution) from the relationship: \( F(x) = \int_{-\infty}^{x} f(s)ds \)

```r
# Standard normal density function f(x) and distribution function F(x)
par(cex = 1.4)
x = seq(-4, 4, length = 200)
plot(x, dnorm(x, mean = 0, sd = 1), type = "l", lwd = 2, ylab = "Density (f)"
plot(x, pnorm(x), type = "l", lwd = 2, ylab = "Cumulative Density (F)"
```
Discrete distributions: replace integrals with sums. $F(x) = \sum_{i=0}^{x} f(i)$

### Poisson (mean 2) density and distribution functions

```r
x = 0:10
par(cex = 1.4)
plot(x, dpois(x, lambda = 2), pch = 19, ylab = "Mass/Density (f)"
plot(x, ppois(x, lambda = 2), type = "s", ylab = "Cumulative Dens. (F)"
points(x, ppois(x, lambda = 2), pch = 19)
```
Exercises

1. Look up which distributions are approximately normal (and for which parameter values), and demonstrate this graphically in R. This may (or may not) be helpful: http://www.math.wm.edu/~leemis/chart/UDR/UDR.html

2. Use the code on previous slides (or your own) and plot the density and distribution functions for these distributions. For each distribution, do this in a 2x2 figure. In the top row, compare to a normal distribution with the same mean and variance. In the bottom row, do the same but with parameters where the normal approximation fails.

3. For the programmers: Too easy? Automate this with a for loop, or use the lattice or ggplot2 packages for the graphics.