Stochastic Models
Week 11 – Monday
Mathematical Modeling (Math 420/620)

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Homogeneous Poisson Process

**Times between events** are exponential (rate $r$):

$$T_k \sim \text{Exponential}(r)$$

**Event times** are Gamma (shape $k$, rate $r$), i.e. $S_k = \sum_{i=0}^{k}$:

$$S_k \sim \text{Gamma}(k, r)$$

Number in interval $[0, T]$ is as a counting process $N_T$ which is **Poisson** distributed with mean $\lambda = r \cdot T$, i.e.

$$P(N_T = n) = \exp(-\lambda) \frac{\lambda^n}{n!}$$
Homogeneous Poisson Process

```
Tk = rexp(15, rate=5); Sk = cumsum(Tk);
plot(Sk, Sk*0, pch="|", xlab="Time (or Space)", yaxt="n", xlim=c(0, max(Sk)));
```
Inhomogeneous Poisson Process

Assumes rate $r$ non-constant, for example

$$r(t) = r_0 + A \sin(\omega t)$$

The Counting Process has mean

$$E(N_T) = \frac{1}{T} \int_0^T \lambda(t) \, dt = \int_0^T r(t) \, dt$$

where we define $\lambda(t) = r(t) T$. 
Inhomogeneous Poisson Process

- Simulated HPP (Pre-thinning; rate=r_{max})
- Thinned Events (IPP with rate=r(t))
Inhomogeneous Poisson Process (Simulation in R)

```r
# First the rate function \( r(t) = r_0 + A \sin(\omega t) \)
r0 = 5; A=4.9; w=1; rmax = r0+A;
r = function(t) { r0 + A*sin(w*t) }

# Simulate at \( r_{\text{max}} = r_0 + A \)
set.seed(1)
Tk = rexp(150,rate=rmax); Sk = cumsum(Tk);

# Thin the events generated at \( r_{\text{max}} \) w.p. \( r(t)/r_{\text{max}} \)
Ps = r(Sk)/rmax; RNs = runif(length(Sk))
keep = (RNs <= Ps); # Element-wise comparison. Returns vector of T/F.
Skthinned = Sk[keep]; # Keep only those with RNs <= Ps

# Plot thinned and remaining events.
plot(Sk,0*Sk,pch="|",col="red", xlab="Time", ylab=""); abline(v=0,h=0)
points(Skthinned,0*Skthinned,pch="|")

# Plot a normalized \( r(t) \) curve
curve(r(x)/rmax, add=TRUE); text(3,0.85,"\( r(t) / r_{\text{max}} \)"
legend("bottom",c("Simulated HPP (Pre-thinning; rate=rmax)",
"Thinned Events (IPP with rate=r(t))"), col=c("red","black"),pch='|')"