Week 2 – Monday

Mathematical Modeling (Math 420/620)

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31 Aug, 2015

MATH COMPETITIONS!

Intermountain Math Competition

November 14, 2015

William Lowell Putnam Mathematical Competition

1st Saturday in December
8-11am, 1-4pm

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Mathematical Modeling

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"Anyone who can understand ... the question as it was presented to you should be able to understand your answer."

–MMM

Example 1.1

Q: A pig weighing 200 pounds gains 5 pounds per day and costs 45 cents a day to keep. The market price for pigs is 65 cents per pound, but is falling 1 cent per day. When should the pig be sold?

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Variables:
$t =$ time (days)
$w =$ weight of pig (lbs)
$p =$ price for pigs (\$/lb)
$C =$ cost of keeping pig t days (\$)
$R =$ revenue obtained by selling pig (\$)
$P =$ profit from sale of pig (\$)

Assumptions:
$w = 200 + 5t$
$p = 0.65 - 0.01t$
$C = 0.45t$
$R = p \cdot w$
$P = R - C$
$t \geq 0$

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Goal: Find the maximum of the profit function $P(t)$.

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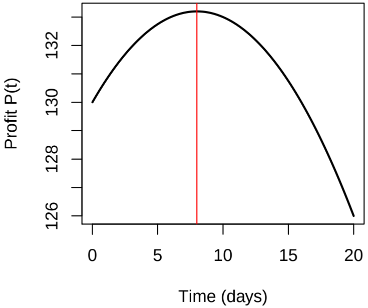
New Q: What value of t maximizes $P(t)$?

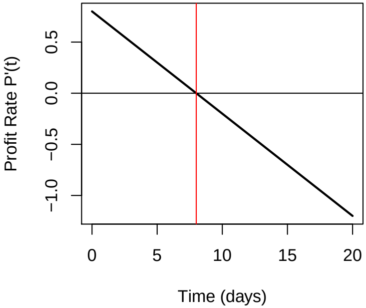
Step 3: Formulate the Model

Profit P is revenue (R) minus cost (C), therefore

$$\begin{aligned} P(t) &= R(t) - C(t) \\ &= p \cdot w - c \cdot t \\ &= (0.65 - 0.01\ t)(200 + 5\ t) - 0.45\ t \end{aligned}$$

```r
## R code to plot P(t) and P'(t)
x=seq(0,20,length=200)
Pt=expression((0.65-0.01*x)*(200+5*x)-0.45*x,'x')
dPt=D(Pt,'x')
plot(x, (0.65-0.01*x)*(200+5*x)-0.45*x, type="l", lwd=2,
        ylab="Profit P(t)", xlab="Time (days)"); abline(v=8, col="red")
plot(x, eval(dPt), type="l", lwd=2, ylab="Profit Rate P'(t)",
        xlab="Time (days)")
abline(h=0); abline(v=8, col="red")
```





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A: To maximize profit, sell the 200 pound pig **8 days** later when it weighs **240 pounds** for a profit of **\$133.20**.

How sensitive is our answer to different inputs?

Let the rate at which the price falls per day be

$$r = 0.10$$

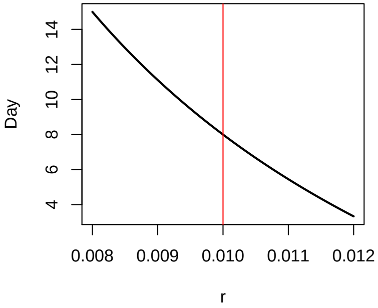
Then our price per pound is

$$p(t) = 0.65 - r\,t$$

Thus, the optimal selling time is the root of

$$P'(t) = \frac{-2(25\,r\,t + 500\,r - 7)}{5}$$

```r
## R code to plot `best time to sell' as function of r, g
r=seq(0.008,0.012,length=100)
plot(r, (7-500*r)/(25*r), type="l", lwd=2, ylab="Day")
abline(v=0.01, col="red")
g=seq(3,7,length=100)
plot(g, 5*(13*g-49)/(2*g), type="l", lwd=2, ylab="Day")
abline(v=5, col="red")
```



(Relative) Sensitivity

Let $T = \arg\max_t P(t)$. Then the proportional change in T per small change in r is approximately

$$\frac{\Delta T/T}{\Delta r/r} \longrightarrow \frac{dT}{dr}\frac{r}{T}$$

Definition: For any response R and parameter p we define the (relative) *sensitivity* of R to p as

$$S(R,p) = \frac{dR}{dp}\frac{p}{R}$$

```r
## Absolute sensitivity of `best time to sell' to r, g
r=seq(0.008,0.012,length=100); Tr=(7-500*r)/(25*r)
plot(r[-1]-diff(r)/2,diff(Tr)/diff(r), type="l", lwd=2, ylab="dT/dr", xlab="r")
abline(v=0.01, col="red")
g=seq(3,7,length=100); Tg = 5*(13*g-49)/(2*g)
plot(g[-1]-diff(g)/2,diff(Tg)/diff(g), type="l", lwd=2, ylab="dT/dg", xlab="g")
abline(v=5, col="red")
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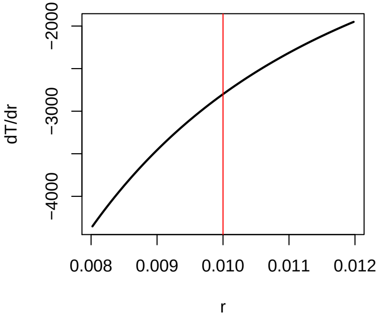

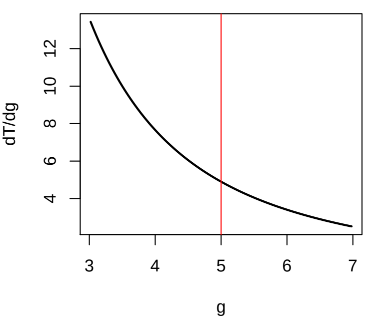

```
## Relative sensitivity of `best time to sell' to r, g
r=seq(0.008,0.012,length=100); Tr=(7-500*r)/(25*r)
plot(r[-1]-diff(r)/2,diff(Tr)/diff(r) * (r[-1]-diff(r)/2)/(Tr[-1]-diff(Tr)/2),
abline(v=0.01, col="red")
g=seq(3,7,length=100); Tg = 5*(13*g-49)/(2*g)
plot(g[-1]-diff(g)/2,diff(Tg)/diff(g) * (g[-1]-diff(g)/2)/(Tg[-1]-diff(Tg)/2),
abline(v=5, col="red")
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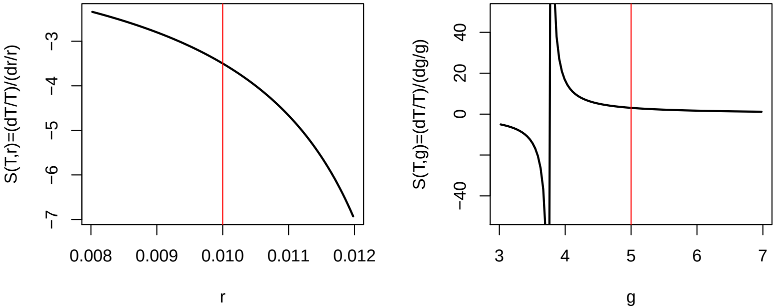

Sensitivity Analysis

A Sensitivity Analysis can

1. ... identify parameter values for which small changes yield large changes in focal output quantities.
2. ... identify the parameters for which larger parameter uncertainty will lead to large output/prediction uncertainty.
3. ... identify where changing a parameter will increase or decrease a given output quantity.