Week 4 – Monday

Mathematical Modeling (Math 420/620)

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14 Sept, 2015

Announcements

Date correction for Undergrad Research workshop at Brown:

The workshop takes place Nov 13, not Nov 15!

More information at:

www.dam.brown.edu/people/lipshutz/workshop2015.html

Apply by **16 Oct** (Cover Letter, CV, 1 Reference Letter) at:

www.mathprograms.org/db/programs/384

Questions?

Multivariable Optimization: Ex 2.1 (Pg 21)

Q: How many 19- & 21-inch TVs maximize profit?

Profit Equation:

$$\begin{aligned} P(x_{19}, x_{21}) = p\,x_{19} + q\,x_{21} - C = & (p_0 - p_{19}\,x_{19} - p_{21}\,x_{21})x_{19} + \\ & (q_0 - q_{19}\,x_{19} - q_{21}\,x_{21})x_{21} - \\ & (c_0 + c_{19}\,x_{19} + c_{21}\,x_{21}) \end{aligned}$$

Time dependent (aka "Variables"):

x_{19} - number of 19in TVs sold

x_{21} - number of 21in TVs sold

p - 19in TV selling price ($)

q - 21in TV selling price ($)

Time independent (aka "Constants"; "Parameters"):

p_0 - Retail price of 19-inch TV

q_0 - Retail price of 19-inch TV

p_{19} - 19in discount / 19in TV sold

p_{21} - 19in discount / 21in TV sold

q_{19} - 21in discount / 19in TV sold

...

Multivariable Optimization: Ex 2.1 (Pg 21)

Approach #1: Find "interior" optima by solving $\nabla P = 0$.

Using Maxima, we can find and solve

$$\begin{aligned}\frac{\partial P}{\partial x_{19}} =& p_0 - 2p_{19}x_{19} - (q_{19} + p_{21})x_{21} - c_{19} = 0 \\ \frac{\partial P}{\partial x_{21}} =& q_0 - (q_{19} + p_{21})x_{19} - 2q_{21}x_{21} - c_{21} = 0\end{aligned}$$

Solving and using the given parameter values yields

$$\mathbf{x_{19} = 4735.04\ldots} \qquad \mathbf{x_{21} = 7042.74\ldots}$$

Multivariable Optimization: Ex 2.1 (Pg 21)

Approach #2: Use generic optimization routines to computationally maximize $P(x_{19}, x_{21})$ over $x_{19}, x_{21} > 0$.

See Ch2-optimization.R

$\uparrow$

Updated!

Constraints

Do we assume **global** or **local** optima?

Ex: Minimum of $(x - r)^2$ over $x \in \mathbb{R}$
Ex: Maximum of $\sin^2(x)$ over $x \in [0, 2\pi]$.

Are there domain constraints?

Box Constraints are the simplest domains restrictions:
Ex: Real-world constraint on sensible parameter values.
Ex: Minimize $f(\vec{x})$ given $l_i \leq x_i \leq u_i$
Ex: (TV Example) Recall we required $x_{19},\ x_{21} > 0$

Remember to check the boundary values!

Constraints

Equality Constraints:

Ex: Maximize $f(x, y)$ w/ constraints $g_i(x, y) = c_i$
Solution: Continuous functions? Use LaGrange Multipliers.

Inequality Constraints:

Ex: Maximize **linear** $f(x, y)$; linear constraints $g_i(x, y) \leq c_i$
Ex: Maximize **quadric** $f(x, y)$; linear constraints $g_i(x, y) \leq c_i$
Solution: Linear and Quadratic Programming, respectively.

LaGrange Multipliers

LaGrange Multipliers arise from necessary conditions for optimizing objective function $f(x)$ w/ **equality constraints** $g_i(x) = c_i$.

Theorem

If f and all g_i are **continuously differentiable**, *then:*

(a) *To maximize f, input x must satisfy*

$$\nabla f(x) = \sum_j \lambda_j \nabla g_j(x).$$

(b) *To minimize f, input x must satisfy*

$$-\nabla f(x) = \sum_j \lambda_j \nabla g_j(x).$$

Generalizing LaGrange Multipliers

Karush-Kuhn-Tucker (KKT) Multipliers arise from necessary conditions for optimizing objective function $f(x)$ w/ **equality** $(g_i(x) = c_i)$ and **ineq. constraints** $(\mathrm{h}_j(x) \leq d_j)$.

Theorem

If f and all g_i and h_j are **continuously differentiable**, *then:*

(a) *To maximize f, input x must satisfy*

$$\nabla f(x) = \sum_i \mu_i \nabla h_i(x) + \sum_j \lambda_j \nabla g_j(x).$$

(b) *To minimize f, input x must satisfy*

$$-\nabla f(x) = \sum_i \mu_i \nabla h_i(x) + \sum_j \lambda_j \nabla g_j(x).$$