EECB Colloquium

High temperatures and the natural history of an impending extinction

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2:30-3:30pm, Thursday (10/1) in DMSC 103

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Dynamic Model (ODE) Basics

Suppose $\mathbf{x} \in \mathbb{R}^n$, functions $f = [f_1, f_2, \ldots, f_n]$ are smooth\(^1\), and

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0.$$  

\(^1\)Continuous partial derivatives near $\mathbf{x}_0$ guarantee existence, uniqueness of solutions.
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State Variables: $x = [x_1, x_2, \ldots, x_n]$
Initial Conditions: $x_0$
State Space: $S \subseteq \mathbb{R}^n$
Vector Field: $f$
Parameter Space: (Ex) $\mathbb{R}^{n^2}$ if $f$ is a full linear system.
Trajectory/Orbit: Solutions $x(t)$ to the above IVP.

See also: order, (non)autonomous, (non)homogenous

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Equilibria

Trajectories are often categorized by qualitative properties (e.g. steady-state vs. cycling vs. chaos) of their asymptotic behavior (i.e., what do solutions look like as \( t \to \infty \)?)

Equilibrium solutions are the natural place to begin studying those asymptotic properties.

**Definition**

An equilibrium of

\[
\frac{dx}{dt} = f(x)
\]

is any constant solution \( x(t) = x_* \) which therefore satisfies

\[
f(x_*) = 0.
\]
Find all equilibrium solutions to each of the following ODEs:

1. \[ \frac{dN}{dt} = r N \]
2. \[ \frac{dx}{dt} = K - x \]
3. \[ \frac{dx}{dt} = x(K - x) \]
4. \[ \frac{dx}{dt} = r x \left(1 - \frac{x}{K}\right) \]
5. \[ \frac{dx}{dt} = x \left(1 - x\right) \left(a - x\right) \]
6. \[ \frac{dx}{dt} = \sin(x) \]
Example 4.1 – Two-species Competition

Goal: When can the two tree species coexist?

\[ \dot{H} = r_H H - a_H H^2 - b_H S H \]
\[ \dot{S} = r_S S - a_S S^2 - b_S H S \]

State Variables: (State Space is non-negative orthant in \( \mathbb{R}^2 \))

\( H(t), S(t) \) - Hardwood & Softwood population size (tons/acre)

Rates: (Units are tons/acre/year)

\[ g_H(t) = r_H H - a_H H^2 \] Hardwood growth rate
\[ g_S(t) = r_S S - a_S S^2 \] Softwood growth rate
\[ c_H(t) = b_H S H \] - Hardwood loss rate
\[ c_S(t) = b_S S H \] - Softwood loss rate

Parameters: intrinsic growth rate \( r_i \), *intraspecific* competition coefficients \( a_i \), and *interspecific* competition coefficient \( b_i \).
Question: Suppose $x_*$ is an equilibrium solution to

$$\frac{dx}{dt} = f(x)$$

and assume we perturb our initial condition to be $\epsilon$-close to that value (i.e., let $x_0 \approx x_*$).

Then does that trajectory converge to (or diverge from, or stay near) the equilibrium value $x_*$?
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Then does that trajectory converge to (or diverge from, or stay near) the equilibrium value $x_*$?

Answer: Conduct a stability analysis of the equilibrium $x_*$. 
Stability Concepts

1. We say $x_*$ is **locally asymptotically stable (LAS)** (or sometimes just *locally stable* or *attracting*) if all nearby trajectories converge to $x_*$ (i.e., $x(t) \to x_*$ as $t \to \infty$).
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4. We call \( x_* \) \textbf{neutrally stable} if it is Lyapunov Stable but not attracting.
Phase Space & 1-D Vector Fields

**Phase Space**: Horizontal axis $x$, vertical axis $\frac{dx}{dt}$.

Bacterial growth model from Hurtado, 2012.

\[
\frac{dp}{dt} = r p (1 - p) - \frac{k p}{\mu + p}
\]

Where

\[
p_{\text{crit}} = \frac{\left((1 - \mu) + \sqrt{(1 + \mu)^2 - \frac{4}{r} k}\right)}{2}
\]

\[
p_{\text{max}} = \frac{\left((1 - \mu) + \sqrt{(1 + \mu)^2 - \frac{4}{r} k}\right)}{2}
\]
Sketch the *phase portrait* for each of the following, and use it to determine the stability of each equilibrium point:

1. \( \frac{dx}{dt} = K - x \)
2. \( \frac{dx}{dt} = x \left( K - x \right) \)
3. \( \frac{dx}{dt} = r \cdot x \left( 1 - \frac{x}{K} \right) \)
4. \( \frac{dx}{dt} = x \left( 1 - x \right) \left( a - x \right) \)
5. \( \frac{dx}{dt} = \sin(x) \)