Dynamical Systems
Week 6 – Monday

Mathematical Modeling (Math 420/620)

Paul J. Hurtado

1 Oct, 2015

EECB Colloquium

High temperatures and the natural history of an impending extinction

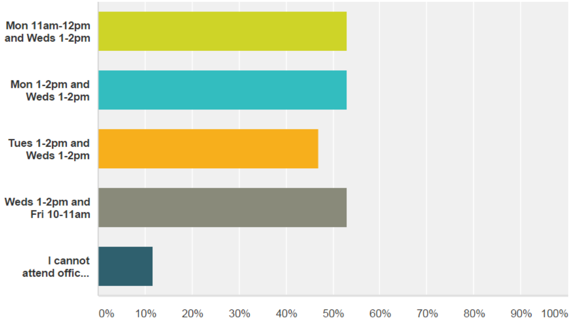
Craig Benkman, University of Wyoming

2:30-3:30pm, **Thursday** (10/1) in **DMSC 103**

www.unr.edu/eecb/colloquium

Which of the following options includes one or more time slots which you could attend office hours in DMS 220?

Answered: 17 Skipped: 0



Answer Choices	Responses	
Mon 11am-12pm and Weds 1-2pm	52.94%	9
Mon 1-2pm and Weds 1-2pm	52.94%	9
Tues 1-2pm and Weds 1-2pm	47.06%	8
Weds 1-2pm and Fri 10-11am	52.94%	9
I cannot attend office hours during any of the times above.	11.76%	2
Total Respondents: 17		

Office Hours

Schedule:

Monday	11am-12pm
Wednesday	1-2pm
Friday	10-11am[1]

[1] These may occasionally be canceled or shortened, so try and let me know in advance if you plan to stop by!

Dynamic Models (ODEs)

Let

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x_0}$$

where $\mathbf{x}(t) \in \mathbb{R}^n \; \forall t \in \mathbb{R}$, and f is smooth.

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Common Question in Applications:
What are the asymptotic dynamics of this model?

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(1) Equilibrium Stability Analysis and
(2) Bifurcation Analysis[2]

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[2] We'll only briefly see bifurcation theory in this course. For more on the subject, I highly recommend Dynamical Systems & Chaos by Steve Strogatz.

Equilibria

Definition

An **equilibrium** of

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$$

is any *constant* solution $\mathbf{x}(t) = \mathbf{x}_*$ which therefore satisfies

$$f(\mathbf{x}_*) = 0.$$

In practice, solutions to most models asymptotically approach an equilibrium point, or other *attractor* (e.g. a limit cycle).

Equilibrium Stability

Theorem

(1D) An **equilibrium** x_* *of* $\dot{x} = f(x)$ *is* **locally asymptotically stable** *if*

$$f'(x_*) < 0$$

and is unstable if

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Sketch of Proof.

If $u = x - x_*$, and f is smooth near x_* then $u = 0$ is an equilbrium of $\dot{u} \approx f'(x_*)\, u$ which has (approximately) exponential solutions that grow away from (or decay towards) 0 depending on the sign of $f'(x_*)$. $\square$

Equilibrium Stability

Theorem

An **equilibrium** $\mathbf{x}_*$ *of* $\dot{\mathbf{x}} = f(\mathbf{x})$ *is* **locally asymptotically stable (LAS)** *if the Jacobian matrix* $\mathbf{J}$ *(where* $J_{ij} = \frac{\delta f_i}{\delta x_j}$*) evaluated at* $\mathbf{x}_*$ *has eigenvalues with negative real parts. That is,* x_* *is LAS if* $Re(\lambda_i) < 0$ *for each of the* n *eigenvalues of matrix* $\mathbf{J}(\mathbf{x}_*)$.

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Sketch of Proof:
Consider the linear approximation of the vector field around $\mathbf{x}_*$. Then for a small neighborhood of $\mathbf{x}_*$,

$$\dot{\mathbf{x}} = f(\mathbf{x}) \approx \mathbf{J}(\mathbf{x}_*)\,\mathbf{x}.$$

Let $\mathbf{u} = \mathbf{x} - \mathbf{x}_*$ and $\mathbf{A} = \mathbf{J}(\mathbf{x}_*)$, then

$$\dot{\mathbf{u}} \approx \mathbf{A}\,\mathbf{u}$$

If $\mathbf{A}$ is full rank then ...

Sketch of Proof (cont'd):
... let $\mathbf{Q}$ be the matrix whose columns are the eigenvectors of $\mathbf{A}$, and let $\mathbf{D} = (\lambda_1, ..., \lambda_n)$. Then doing a standard change-of-coordinates

$$\begin{aligned}\dot{\mathbf{u}} &= \mathbf{Q}\,\mathbf{D}\,\mathbf{Q}^{-1}\,\mathbf{u} \\ \mathbf{Q}^{-1}\,\dot{\mathbf{u}} &= \mathbf{D}\,\mathbf{Q}^{-1}\,\mathbf{u} \\ \dot{\mathbf{y}} &= \mathbf{D}\,\mathbf{y}\end{aligned}$$

which implies $\dot{y}_i = \lambda_i\, y_i$ and thus

$$y_i(t) = y_i(0)\,\exp(\lambda_i\, t).$$

Therefore, **trajectories that begin sufficiently close to equilibrium $\mathbf{x}_*$ will approximately grow or decay at rate $Re(\lambda_i)$ along the corresponding eigenvectors of $\mathbf{J}(\mathbf{x}_*)$.**

Equilbrium Stability

Find all equilibrium solutions to each of the following ODEs:

1. $\dfrac{dx}{dt} = K - x$
2. $\dfrac{dx}{dt} = x\,(1-x)(a-x)$

Example 4.1 – Two-species Competition

$$\dot{H} = r_H\, H - a_H\, H^2 - b_H\, S\, H$$

$$\dot{S} = r_S\, S - a_S\, S^2 - b_S\, H\, S$$

Predator-Prey

$$\dot{x} = r\, x\,(1 - x) - \frac{a\, x\, y}{k + x}$$

$$\dot{y} = \frac{a\, x\, y}{k + x} - y$$