MATH 420 Homework #4 — Due: Weds 9/30

Instructions: A printed copy of your homework should be handed in at **the start of class** the day it is due. Supplementary electronic files (e.g. R scripts or wxMaxima files;) should be emailed to the instructor prior to class and named according to the format LASTNAME-HWX.EXT (ex: Hurtado-HW2.R).

1. In last week's homework, it was given that the 1-dimensional linear ODE

$$\frac{dN}{dt} = \lambda\, N$$

has the solution

$$N(t) = N_0 \exp(\lambda\, t).$$

Assuming $N \neq 0$, this can be shown by the separation of variables method from calculus. Dividing both sides by N and integrating over time interval $[0, t]$ yields

$$\begin{aligned} \int_0^t \frac{dN/dt}{N} dt &= \int_0^t \lambda\, dt \\ \log(N(t)) - \log(N_0) &= \int_0^t \lambda\, dt, \quad \text{where } N_0 \equiv N(0) \\ \log(N(t)) - \log(N_0) &= \lambda\, t \end{aligned}$$

and by adding $\log(N_0)$ to each side and exponentiating, we get

$$N(t) = N_0\, \exp(\lambda\, t)$$

It is rare, in practice, that we can explicitly solve ODE models and find analytical solutions. However, it is a possibility that should not be overlooked! For example, some (e.g., see `http://press.princeton.edu/chapters/s03_8709.pdf`, pg 99, eq. 3.4.5) have claimed that the theta-logistic model (below) has no closed-form solution for $N(t)$.

$$\frac{dN}{dt} = r\, N \left(1 - \left(\frac{N}{K}\right)^\theta\right) \tag{1}$$

Prove this claim false by using separation of variables, as above, to find an expression for $N(t)$ that is a solution to the theta-logistic equation (eq. (1)).

(**Hint:** Look up the steps to finding a solution to the logistic equation, and mimic those. After moving all of the N terms to one side of the equality, do a partial fractions decomposition then integrate. Recall what the derivative $\frac{d}{dx}\log(f(x))$ looks like, and use Maxima as needed.)

2. In class (see website for slides) we discussed the concepts of **state space** and **parameter space** for dynamical systems models. We can also categorize differential equations models by whether they are **linear** or **non-linear**, and by whether they are **autonomous** or **non-autonomous**. Look up the definitions of these terms, and for each model below, give (i) the dimension of their state space, (ii) the dimension of their parameter space, (iii) whether they are linear or non-linear, and (iv) whether they are autonomous or non-autonomous.

   (a) $N(t) \in \mathbb{R}_0^+$ for all t, $r \in \mathbb{R}$.

$$\frac{dN(t)}{dt} = r\,N(t)$$

   (b) $u(\tau) \in \mathbb{R}_0^+$ for all τ.

$$\frac{du}{d\tau} = (1-u)\,u$$

   (c) $N(t) \in \mathbb{R}_0^+$ for all t; $r \in \mathbb{R}^+$, $K(t)$ a positive-valued differentiable function.

$$\frac{dN}{dt} = r\,x\,(1 - N/K(t))$$

   (d) As above, with parameters K_∞, $\theta \in \mathbb{R}^+$.

$$\begin{aligned}\frac{dN}{dt} =& r\,N\left(1-(N/K)^\theta\right)\\ \frac{dK}{dt} =& K_\infty - K\end{aligned}$$

   (e) $x(t) \in \mathbb{R}_0^+$ for all t; m constant.

$$\frac{dx}{dt} = m^3\,x$$