Nondimensionalization Example

Here we nondimensionalize the system of equations

\[
\begin{align*}
\dot{N}_1 &= r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2 \\
\dot{N}_2 &= r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2.
\end{align*}
\]

Note that there are two state variables, one time variable, and six parameters. Scaling both state variables and time should reduce the number of parameters by three.

Writing out the units of each symbol (here we use \(i, j\) to denote 1,2 or 2,1), we have

- \(r_i\) per unit time
- \(N_i\) # individuals
- \(K_i\) # individuals
- \(b_i\) per unit time, per # of individuals

The easiest way to non-dimensionalize \(N_i\) is to scale them each by \(K_i\), therefore divide both sides of our equations by \(K_i\) and gather all \(N_i/K_i\) terms, substituting \(n_i = N_i/K_i\) to get

\[
\dot{n}_i = r_i n_i (1 - n_i) - B_i n_i n_j
\]

We multiplied the last term by \(K_j/K_j\) to scale both \(N_1 \cdot N_2\), thus we defined \(B_i = b_i k_j\).

We now have the intermediate quantities

- \(r_i\) per unit time
- \(n_i\) unitless (interpret as proportion of individuals relative to \(K_i\))
- \(B_i\) per unit time

Next, we scale time by one of the remaining rates, and choose one of the \(r_i\) rates because these will lead to easier interpretation than using say \(B_1\) in an application context. Thus, dividing both equations by \(r_1\) yields

\[
\begin{align*}
\frac{dn_1}{r_1 \, dt} &= n_1 (1 - n_1) - \frac{B_{12}}{r_1} n_1 n_2 \\
\frac{dn_2}{r_1 \, dt} &= \frac{r_2}{r_1} n_2 (1 - n_2) - \frac{B_{21}}{r_1} n_1 n_2
\end{align*}
\]

Set \(\alpha = \frac{B_{12}}{r_1}\), \(\beta = \frac{B_{21}}{r_1}\), \(\gamma = \frac{r_2}{r_1}\) and \(\tau = tr_1\). Then by the linearity of the derivative \(r_1 \, dt = d\tau\) and we have two dimensionless equations with three parameters (reduced from six!):

\[
\begin{align*}
\frac{dn_1}{d\tau} &= n_1 (1 - n_1) - \alpha n_1 n_2 \\
\frac{dn_2}{d\tau} &= \gamma n_2 (1 - n_2) - \beta n_1 n_2.
\end{align*}
\]