Dynamical Systems: Introduction
Mathematical Modeling (Math 420/620)

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Overview

Building Dynamic Models, ODEs

- Mean Field Equations & “Bathtub” Models

Analysis of Dynamic Models (Topic Overview)

- State Space & Vector Fields
- Asymptotic Behavior: What happens as $t \to \infty$? Parameter dependence?
- Equilibrium Stability Analysis
- Other dynamics? Bifurcation Theory
- Other Attractors: Limit Cycles, etc.
- Sensitivity Analysis & Simulation
Example: Exponential Decay

## Ex: tracking atoms experiencing radioactive decay

```r
Ts = sort(rexp(50, 1/100))
Time = seq(0, max(Ts), length=300)
N = Time*0;  # counts of atoms at time t go here.
N[1] = 50;
for(i in 2:300) { N[i] = sum(Ts > Time[i]) }  # number not yet decayed
plot(Time, N); curve(50*(exp(-x/100)), 0, max(Ts), add=TRUE, col="red")
```
Example: Exponential Decay

```r
## Ex: tracking atoms experiencing radioactive decay
Ts = sort(rexp(1e4, 1/100))
Time = seq(0, max(Ts), length=300)
N = Time*0; # counts of atoms at time t go here.
N[1] = 1e4;
for(i in 2:300) { N[i] = sum(Ts > Time[i]) } # number not yet decayed
plot(Time, N); curve(1e4*(exp(-x/100)), 0, max(Ts), add=TRUE, col="red")
```

![Graph showing exponential decay](image-url)
Example: Exponential Decay as Stochastic Map

Simulate as a discrete map with a Binomial # of atoms decaying each time step, i.e.,

\[ N(t + dt) = N(t) - \text{rbinom}(1, n = N(t), \text{prob} = r \cdot dt) \]

```r
N0=1e3; dt=1/100; r=1; N=c(N0); i=1;
while(N[i] > 0) { N[i+1]=N[i]-rbinom(1,N[i],r*dt); i=i+1; } # number not yet decayed
Time=dt*(1:length(N))-dt; plot(Time,N,xlab="Time");
curve(N0*(exp(-r*x)),0,dt*length(N),add=TRUE,col="red")
```
Implicit Assumptions?

Which (implicit) assumptions were made? Which could be relaxed?

Spatially structured interactions?

Small $N$ vs $N \to \infty$?

Time-dependent or $N$-dependent rate?

Others?

Good rule of thumb with ODE models:
Implicit assumptions typically ignore spatial interactions, stochastic variation and/or small numbers of individuals, and/or the discrete nature of individuals.
**Mean Field Equations: Applying LLN, CLT**

**Example:** Suppose there are $N_0$ atoms of radioactive $^{238}_{92}$U. Over time interval $\Delta t$ each can decay w.p. $\lambda \Delta t$.

Let $N(t)$ be the number of uranium atoms. The number lost during time interval $[t, t + \Delta t]$ is approximately a binomial random variable with parameters $n = N(t)$ and $p = \lambda \Delta t$. Thus, the expected number lost is $n p = \lambda N(t) \Delta t$.

Assuming $N_0$ is large, then the Law of Large Numbers (LLN) allows us to claim $N(t + \Delta t) - N(t) \approx -\lambda N(t) \Delta t$. Taking $\Delta t \to 0$ we can derive the mean field model:

$$\frac{dN(t)}{dt} = -\lambda N(t), \quad N(0) = N_0$$
Mean Field Equations

**Example:** Suppose there are $U_0$ atoms of radioactive $^{238}_{92}$U. Over time interval $\Delta t$ each can decay w.p. $\lambda_\alpha \Delta t$ to $^{234}_{90}$Th and $\alpha$ particle $^4_2$He. Thorium-234 can then decay via loss of a $\beta$ particle (positron) to protactinium-234 w.p. $\lambda_\beta \Delta t$.

Let $T(t)$ be the number of thorium atoms, and $P(t)$ the number of protactinium atoms. We can now use the model

$$\frac{dU(t)}{dt} = -\lambda_\alpha U(t)$$

$$\frac{dT(t)}{dt} = \lambda_\alpha U(t) - \lambda_\beta T(t)$$

$$\frac{dP(t)}{dt} = \lambda_\beta T(t)$$
Exercise

Derive the following UTP model

\[
\frac{dU(t)}{dt} = -\lambda_\alpha U(t)
\]
\[
\frac{dT(t)}{dt} = \lambda_\alpha U(t) - \lambda_\beta T(t)
\]
\[
\frac{dP(t)}{dt} = \lambda_\beta T(t)
\]

1. Write a discrete time map (step size \(\Delta t\)) that models the numbers of atoms transitioning states in each time step using Binomial distributions.

2. Use the LLN to find the corresponding mean-field map.

3. Take the limit as \(\Delta t \to 0\) to find the continuous time (ODE) approximation of this mean-field discrete map.
ODEs: “Bathtub” Models

Model the “flow” of mass from one compartment to another:

\[
\frac{dU(t)}{dt} = -\lambda_\alpha U(t)
\]

\[
\frac{dT(t)}{dt} = \lambda_\alpha U(t) - \lambda_\beta T(t)
\]

\[
\frac{dP(t)}{dt} = \lambda_\beta T(t)
\]
Intuition for ODE model terms

- Recall the 5-step process!
  - Question?  Assumptions?  Simplify, etc...

- ODE models often average over heterogeneity, space, etc.

- **Linear terms** correspond to exponential decay rates.

- More complex transition rates? Derive\(^1\) terms accordingly.

\(^1\)Remember: Lie, Cheat, Steal! (see Ch. 9 in Ellner & Guckenheimer)
Dynamic Model (ODE) Basics

Suppose $x \in \mathbb{R}^n$, functions $f = [f_1, f_2, \ldots, f_n]$ are smooth$^2$, and

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0.$$

State Variables: $x = [x_1, x_2, \ldots, x_n]$
Initial Conditions: $x_0$
State Space: $S \subseteq \mathbb{R}^n$ ($n = \# \text{ of state var.}$)
Vector Field: $f$
Parameter Space: Ex: $\mathbb{R}^{n^2}$ for a full linear system.
Trajectory/Orbit: Solutions $x(t)$ to the above IVP.

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$^2$Continuous partial derivatives near $x_0$ guarantee existence, uniqueness of solutions.
Examples

What are the state variables? State space? Parameter space?

1. \[ \frac{dx}{dt} = r x \left( 1 - x/K \right) \]

2. \[ \frac{dN}{dt} = r N \left( 1 - (N/K)^\theta \right) \]

3. \[ \frac{du}{d\tau} = u \left( 1 - u^\theta \right) \]

4. \[ \dot{H} = r_H H - a_H H^2 - b_H S H \]
   \[ \dot{S} = r_S S - a_S S^2 - b_S H S \]
Equilibria

Trajectories are often categorized by **qualitative properties** (e.g. steady-state vs. cycling vs. chaos) of their **asymptotic behavior** (i.e., what do solutions look like as $t \to \infty$?).

Equilibrium solutions are the natural place to begin studying those asymptotic properties.

**Definition**

An **equilibrium** of

\[
\frac{dx}{dt} = f(x)
\]

is any **constant** solution $x(t) = x_*$ which therefore satisfies

\[
f(x_*) = 0.
\]
Find all equilibrium solutions to each of the following ODEs:

1. \[ \frac{dN}{dt} = rN \]
2. \[ \frac{dx}{dt} = K - x \]
3. \[ \frac{dx}{dt} = x(K - x) \]
4. \[ \frac{dx}{dt} = r x \left(1 - \frac{x}{K}\right) \]
5. \[ \frac{dx}{dt} = x \left(1 - x\right)(a - x) \]
6. \[ \frac{dx}{dt} = \sin(x) \]
Stability Concepts

1. We say $\mathbf{x}_*$ is **locally asymptotically stable (LAS)** (or sometimes just **locally stable** or **attracting**) if all nearby trajectories converge to $\mathbf{x}_*$ (i.e., $\mathbf{x}(t) \to \mathbf{x}_*$ as $t \to \infty$).
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3. We say $x_*$ is **Lyapunov Stable** if trajectories that start near it stay near $x_*$. 

4. We call $x_*$ **neutrally stable** if it is Lyapunov Stable but not attracting.
**Phase Space & 1-D Vector Fields**

**Phase Space**: Horizontal axis $x$, vertical axis $\frac{dx}{dt}$.

Bacterial infection growth model from Hurtado, 2012.

\[
\frac{dp}{dt} = r \ p \ (1 - p) - \frac{k \ p}{\mu + p}.
\]

In the figures,

\[
p_{\text{crit}} = \frac{\left( (1 - \mu) + \sqrt{(1 + \mu)^2 - \frac{4}{r}k} \right)}{2},
\]

\[
p_{\text{max}} = \frac{\left( (1 - \mu) + \sqrt{(1 + \mu)^2 - \frac{4}{r}k} \right)}{2}.
\]
Equilibrium Stability

Theorem

(1D) An equilibrium \( x_* \) of \( \dot{x} = f(x) \) is locally asymptotically stable if
\[
f'(x_*) < 0
\]
and is unstable if
\[
f'(x_*) > 0.
\]
Theorem

(1D) An equilibrium $x_*$ of $\dot{x} = f(x)$ is locally asymptotically stable if

$$f'(x_*) < 0$$

and is unstable if

$$f'(x_*) > 0.$$ 

Sketch of Proof.

If $u = x - x_*$, and $f$ is smooth near $x_*$ then $u = 0$ is an equilibrium of $\dot{u} \approx f'(x_*) u$ which has (approximately) exponential solutions that grow away from (or decay towards) 0 depending on the sign of $f'(x_*)$. 

Phase Space & 1-D Vector Fields

Sketch the *phase portrait* for each of the following, and use it to determine the stability of each equilibrium point:

1. \( \frac{dx}{dt} = K - x \)
2. \( \frac{dx}{dt} = x (K - x) \)
3. \( \frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right) \)
4. \( \frac{dx}{dt} = x (1 - x)(a - x) \)
5. \( \frac{dx}{dt} = \sin(x) \)
Equilibrium Stability

**Theorem**

An equilibrium $\mathbf{x}_*$ of $\dot{\mathbf{x}} = f(\mathbf{x})$ is locally asymptotically stable (LAS) if the Jacobian matrix $\mathbf{J}$ (where $J_{ij} = \frac{\delta f_i}{\delta x_j}$) evaluated at $\mathbf{x}_*$ has eigenvalues with negative real parts. That is, $\mathbf{x}_*$ is LAS if $\text{Re}(\lambda_i) < 0$ for each of the $n$ eigenvalues of matrix $\mathbf{J}(\mathbf{x}_*)$. 

**Sketch of Proof:** Consider the linear approximation of the vector field around $\mathbf{x}_*$. Then for a small neighborhood of $\mathbf{x}_*$, $\dot{\mathbf{x}} = f(\mathbf{x}) \approx \mathbf{J}(\mathbf{x}_*) \mathbf{x}$. Let $\mathbf{u} = \mathbf{x} - \mathbf{x}_*$ and $\mathbf{A} = \mathbf{J}(\mathbf{x}_*)$, then $\dot{\mathbf{u}} \approx \mathbf{A} \mathbf{u}$. If $\mathbf{A}$ is full rank then ...
Equilibrium Stability

**Theorem**

An equilibrium $x_\star$ of $\dot{x} = f(x)$ is **locally asymptotically stable (LAS)** if the Jacobian matrix $J$ (where $J_{ij} = \frac{\delta f_i}{\delta x_j}$) evaluated at $x_\star$ has eigenvalues with negative real parts. That is, $x_\star$ is LAS if $\text{Re}(\lambda_i) < 0$ for each of the $n$ eigenvalues of matrix $J(x_\star)$.

**Sketch of Proof:**

Consider the linear approximation of the vector field around $x_\star$. Then for a small neighborhood of $x_\star$,

$$\dot{x} = f(x) \approx J(x_\star) x.$$

Let $u = x - x_\star$ and $A = J(x_\star)$, then

$$\dot{u} \approx Au$$

If $A$ is full rank then ...
Sketch of Proof (cont’d):

... let \( Q \) be the matrix whose columns are the eigenvectors of \( A \), and let \( D = (\lambda_1, ..., \lambda_n) \). Then doing a standard change-of-coordinates

\[
\dot{u} = Q D Q^{-1} u \\
Q^{-1} \dot{u} = D Q^{-1} u \\
\dot{y} = D y
\]

which implies \( \dot{y}_i = \lambda_i y_i \) and thus

\[
y_i(t) = y_i(0) \exp(\lambda_i t).
\]

Therefore, trajectories that begin sufficiently close to equilibrium \( x_* \) will approximately grow or decay at rate \( Re(\lambda_i) \) along the corresponding eigenvectors of \( J(x_*) \).
Equilibrium Stability

Find all equilibrium solutions to each of the following ODEs:

1. \( \frac{dx}{dt} = K - x \)
2. \( \frac{dx}{dt} = x (1 - x)(a - x) \)
Two-species Competition (MMM Ex. 4.1)

\[
\dot{H} = r_H H - a_H H^2 - b_H S H
\]

\[
\dot{S} = r_S S - a_S S^2 - b_S H S
\]

Predator-Prey

\[
\dot{x} = r x (1 - x) - \frac{a x y}{k + x}
\]

\[
\dot{y} = \frac{a x y}{k + x} - y
\]
Two-species Competition

**Goal:** When can the two tree species coexist?

\[
\begin{align*}
\dot{H} &= r_H H - a_H H^2 - b_H S H \\
\dot{S} &= r_S S - a_S S^2 - b_S H S
\end{align*}
\]

**State Variables:** *(State Space is non-negative orthant in \( \mathbb{R}^2 \))*

- \( H(t), S(t) \) - Hardwood & Softwood population size (tons/acre)

**Rates:** (Units are tons/acre/year)

- \( g_H(t) = r_H H - a_H H^2 \) Hardwood growth rate
- \( g_S(t) = r_S S - a_S S^2 \) Softwood growth rate
- \( c_H(t) = b_H S H \) - Competitive impact on Hardwoods
- \( c_S(t) = b_S S H \) - Competitive impact on Softwoods

**Parameters:** intrinsic growth rate \( r_i \), *intraspecific* competition coefficients \( a_i \), and *interspecific* competition coefficient \( b_i \).
Overview: Dynamic Models (ODEs)

Let
\[ \frac{dx}{dt} = f(x), \quad x(0) = x_0 \]

where \( x(t) \in \mathbb{R}^n \) \( \forall t \in \mathbb{R} \), and \( f \) is smooth.

**Common Question in Applications:**
What are the asymptotic dynamics of this model?

**Approach:**
(1) Equilibrium Stability Analysis and
(2) Bifurcation Analysis\(^3\)

\(^3\)We'll only briefly see bifurcation theory in this course. For more on the subject, I highly recommend *Dynamical Systems & Chaos* by Steve Strogatz.