MATH 461/661 Homework 9

Due Thursday, April 23, 2015 at the beginning of lecture.

In order to receive full credit, you must (1) present the solutions in the order of the assigned problems, (2) write or type legibly, and (3) justify your answers appropriately. For problems with answers given in the back of the text, please remember to **show your work!**

Note: To save time, you may evaluate integrals or simplify sums using a **symbolic calculator**, software such as **Maxima**, **Maple** or **Mathematica** (available in some UNR computer labs and via citrix.unr.edu), or free online resources like **Wolfram Alpha** (www.wolframalpha.com) and **Sage Notebook** (sagenb.com). Please note where you use such resources in your answers.

1. **3.9.10** *Hint: Don't use integrals!* Instead, use properties of $E(\cdot)$, the definition of variance, & the mean $\left(\frac{a+b}{2}\right)$ and variance $\left(\frac{(b-a)^2}{12}\right)$ of a uniform r.v. on $[a, b]$.

2. **(Memoryless property exponential waiting times):** Let T be an exponential r.v. with rate r (i.e., $f_T(t) = r\exp(-rt)$ and $F_T(t) = P(T \leq t) = 1 - \exp(-rt)$. T represents the waiting time until an event occurs. **Suppose 10 minutes go by, and no event occurs. What's the distribution of the waiting time after the 10 minute mark?** To answer this, first define U to be the waiting time after 10 minutes, where $F_U(t) = P(U \leq t) = P(T \leq 10 + t | T \geq 10)$. Use the cdf of T, and the definition of conditional probability, to show that U is exponential with rate r. That is, show that

$$P(U \leq t) = 1 - \exp(-rt).$$

3. See example 3.12.3 (pg 208) which shows that the MGF for T, an exponential r.v. with rate r, is $M_T(t) = \frac{r}{r-t}$. See Theorem 4.6.5 (pg 273) which shows that the MGF for Y, a gamma r.v. with shape parameter n and rate r, is $M_Y(t) = \frac{r^n}{(r-t)^n}$. Use the properties of moment generating functions spelled out in Theorems 3.12.2 and 3.12.3 (pg 214) to **show that the sum of n *iid* exponentials (with rate r) is a Gamma distributed r.v.**

4. Use Theorem 4.6.5 to provide an alternate proof to Theorem 4.6.4 (pg 273) using MGFs.

5. Let X_1,..., X_n be independent exponential r.v.s, with their own (different) rates r_i.

   a. Using the MGFs from #3 above, find $M_{X_i}(0)$, ${M_{X_i}}'(0)$ and $E(X_i)$.

   b. Let $Y = \sum_i X_i$. Using the properties of MGFs, and the generalized product rule for the product of n differentiable functions $G_i(t)$,

$$\frac{d}{dt}\left(\prod_{i=1}^{n} G_i(t)\right) = \sum_{i=1}^{n}\left(G_i'(t)\prod_{j\neq i} G_j(t)\right)$$

   find $M_Y'(0)$ to confirm the obvious relationship $E(Y) = \sum_i E(X_i)$

6. **(Order Statistics)** Let $X_1, \ldots, X_n$ be iid exponential random variables with $E(X_i) = 1$. Define $X_{\max} = \max(X_1, ..., X_2)$. If $n = 10$, what is the probability $P(X_{\max} > 5)$? Hint: Use Theorem 3.10.1 on pg 194.

7. **4.2.17**

8. **4.2.22 (461)**

   4.2.25 (661)

9. **4.3.2(b)**

10. **4.3.5(d)**

11. **4.4.7**

12. **4.5.6** (To clarify, "the number of trials in excess of r" is just the number of failures before the rth success.)

EXTRA CREDIT: 4.3.3, 4.3.15, 4.6.6