## COMBINATORICS: counting, ordering, arranging

**Multiplication Rule:** If operation A can be performed in n different ways and operation B can be performed in m different ways, then the sequence of these two operations (say, AB) can be performed in $n \cdot m$ ways.

**Extension of Multiplication Rule to k operations:** If operations $A_i$, $i = 1, \ldots, k$ can be performed in $n_i$ different ways, then the ordered sequence (operation $A_1$, operation $A_2$, $\ldots$, operation $A_k$) can be performed in $n_1 n_2 \cdots n_k$ ways.

**Permutations:** An arrangement of k objects in a row is called *a permutation of length k.*

**Number of permutations of k elements chosen from a set on n elements:** The number of permutations of length k, that can be formed from a set of n distinct objects is

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}. \tag{1}$$

**Number of permutations of n elements chosen from a set on n elements:** The number of permutations of length n (ordered sequences of length n), that can be formed from a set of n distinct objects is

$$n(n-1)(n-2)\cdots(1) = n!. \tag{2}$$

**Approximation for n! (Stirling's Formula):** $n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}$.

**Number of permutations of elements that are not all different:** The number of permutations of length n, that can be formed from a set of $n_1$ objects of type 1, $n_2$ objects of type 2, $\ldots$, $n_k$ objects of type k, where $\sum_{i=1}^{k} n_i = n$, is

$$\frac{n!}{n_1! n_2! \cdots n_k!}. \tag{3}$$

**Combinations:** A set of k unordered objects is called *a combination of size k.*

**Number of combinations of size k of n distinct objects:** The number of ways to form combinations of size k from a set of n distinct objects, no repetitions, is denoted by the Newton symbol (or *binomial coefficient*) $\binom{n}{k}$ , and equal to

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{4}$$

**NOTE:** The number of combinations of size k of n distinct objects is the number of different subsets of size k formed from a set of n elements.

**Note:** Not all counting problems are straight foreward combinations or permutations. Often in those situations, we can still count things correctly by breaking the counting problem into cases that are more easily countable.

**Ex:** Consider the set of letters with an extra $A$: $\{A, A, B, C, .., Z\}$. How many unique combinations of four letters can be drawn from this set?

**Answer:** We can partition all the sequences of four letters into three cases: (1) no $A$s, (2) one $A$, or (3) both $A$s, and consider them distinct (e.g., as $A_1$ and $A_2$) where it might aid in counting. How many are in each group?

(1) If there are no $A$s, there are $\binom{25}{4}$ sequences.

(2) Treating each $A$ as distinct, for each sequence with only $A_1$ there is an otherwise identical sequence with $A_2$, so we'd like to half the count of these sequences where the $A$s are treated as distinct. Alternatively, treating the $A$ as indistinct, we simply count ways of choosing the other three letters: $\binom{25}{3}$.

(3) For unordered sets that contain both, order doesn't matter, so these can counted as above: There are $\binom{25}{2}$ of these sets.

Therefore, we have $\binom{25}{4} + \binom{25}{3} + \binom{25}{2}$ unique subsets of size 4 that can be drawn from the set of letters $\{A, A, B, C, ..., Z\}$.

**Combinatorial probabilities - classical definition of probability:** Suppose there are n simple outcomes in a sample space S. Let event A consist of m of those outcomes. Suppose also that all outcomes are equally likely. Then, the probability of event A is defined as P(A)=m/n.