EXPECTED VALUES OF RANDOM VARIABLES

To get an idea about the *central tendency* for a random variable, we compute its **expected value** (mean).

Definition Let X be a random variable.

1. If X is a discrete random variable with pdf $p_X(k)$, then the expected value of X is given by
$$E(X) = \mu = \mu_X = \sum_{\text{all k}} k \cdot p_X(k) = \sum_{\text{all k}} k \cdot P(X = k)$$
2. If X is a continuous random variable with pdf f, then
$$EX = \mu = \mu_X = \int_{-\infty}^{\infty} x f(x) dx.$$
3. If X is a mixed random variable with cdf F, then the expected value of X is given by
$$E(X) = \mu = \mu_X = \int_{-\infty}^{\infty} x F'(x) dx + \sum_{\text{all k}} k \cdot P(X = k),$$
where F' is the derivative of F where the derivative exists and k's in the summation are the "discrete" values of X.

NOTE: For the expectation of a random variable to exist, we assume that all integrals and sums in the definition of the expectation above converge **absolutely**.

Median of a random variable - a value "dividing the distribution of X in halfs. If X is a discrete random variable, then its median m is the point for which $P(X < m) = P(X > m)$. If there are two values m and m' such that $P(X \leq m) = 0.5$ and $P(X \geq m') = 0.5$, the median is the average of m and m', $(m + m')/2$.

If X is a continuous random variable with pdf f, the median is the solution of the equation:
$$\int_{-\infty}^{m} f(x) dx = 0.5.$$

EXPECTED VALUES OF A FUNCTION OF A RANDOM VARIABLE

Theorem. Let X be a random variable. Let $g(\cdot)$ be a function of X.
If X is discrete with pdf $p_X(k)$, then the expected value of $g(X)$ is given by
$$Eg(X) = \sum_{\text{all k}} g(k) \cdot p_X(k) = \sum_{\text{all k}} g(k) \cdot P(X = k),$$

provided that $\sum_{\text{all k}} \mid g(k) \mid p_X(k)$ is finite.

If X is a continuous random variable with pdf $f_X(x)$, and if g is a continuous function, then the expected value of $g(X)$ is given by

$$Eg(X) = \int_{-\infty}^{\infty} g(x)f(x)dx,$$

provided that $\int_{-\infty}^{\infty} \mid g(x) \mid f(x)dx$ is finite.

NOTE: Expected value is a linear operator, that is $E(aX + b) = aE(X) + b$, for any rv X.

PROPERTIES OF E$(\cdot)$

1. Linearity: $E(aX + b) = aE(X) + b$, or in general, $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X)$
2. For an indicator function, $E(\mathbb{1}_A(X)) = P_X(A)$
3. For X a finite random variable, $S = \{1, ..., n\}$, then
$$E(X) = \sum_{j=1}^{n} P(X \geq j)$$
4. (*Markov Inequality*) For $X \geq 0$,
$$P(X \geq a) \leq \frac{E(X)}{a}$$
5. If X and Y are independent, $E(XY) = E(X)E(Y)$.

VARIANCE OF A RANDOM VARIABLE

To get an idea about variability of a random variable, we look at the *measures of spread.* These include **variance and standard deviation**.

Definition. Variance of a random variable, denoted Var(X) or σ^2, is the average of its squared deviations from the mean μ. Let X be a random variable.

1. If X is a discrete random variable with pdf $p_X(k)$ and mean μ_X, then the variance of X is given by
$$Var(X) = \sigma^2 = E[(X - \mu_X)^2] = \sum_{\text{all k}} (k - \mu_X)^2 p_X(k) = \sum_{\text{all k}} (k - \mu_X)^2 P(X = k)$$

2. If X is a continuous random variable with pdf f and mean μ_X, then

$$Var(X) = \sigma^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)dx.$$

3. If X is a mixed random variable with cdf F and mean μ_X, then the variance of X is given by

$$Var(X) = \sigma^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 F'(x)dx + \sum_{\text{all k}} (k - \mu_X)^2 P(X = k),$$

where F' is the derivative of F where the derivative exists and k's in the summation are the "discrete" values of X.

NOTE: If EX^2 is not finite, then variance does not exist.

Definition. Standard deviation σ of a r.v. X is square root of its variance (if exists): $\sigma = \sqrt{Var(X)}$.

NOTE: The units of variance are square units of the random variable. The units of standard deviation are the same as the units of the random variable.

Theorem: Let X be a random variable with variance σ^2. Then, we can compute σ^2 as follows:

$$Var(X) = \sigma^2 = E(X^2) - \mu_X^2 = E(X^2) - [E(X)]^2$$

Theorem: Let X be a r.v. with variance σ^2. Then variance of $aX + b$, for any real a and b, is given by:

$$Var(aX + b) = a^2 Var(X).$$

HIGHER MOMENTS OF A RANDOM VARIABLE

Expected value is called the *first moment* of a random variable. Variance is called the *second central moment* or *second moment about the mean* of a random variable. In general, we have the following definition of the central and ordinary moments of random variables.

Definition: Let X be an r.v. Then the

1. The r^th moment of X (about the origin) is $\mu_r = E(X^r)$, provided that the moment exists.
2. The r^th moment of X about the mean is $\mu_r' = E[(X - \mu_X)^r]$, provided that the moment exists.