JOINT DENSITIES - RANDOM VECTORS

Joint densities describe probability distributions of *random vectors*. A random vector $\mathbf{X}$ is an n-dimensional vector where each component is itself a random variable, i.e., $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, where all $X_i's$ are rvs.

Discrete random vectors are described by the *joint probability density function of* X_i (or joint pdf), $i = \{1, \cdots, n\}$ denoted by

$$P(X = x) = P(s \in S : X_i(s) = x_i \text{ for all } i) = p_X(x_1, .., x_n)$$

Another name for the joint pdf of a discrete random vector is *joint probability mass function (pmf).*

Computing probabilities for discrete random vectors. For any subset A of R^2, we have

$$P((X,Y) \in A) = \sum_{(x,y)\in A} P(X = x, Y = y) = \sum_{(x,y)\in A} p_{X,Y}(x, y).$$

Continuous random vectors are described by the *joint probability density function of X and Y (or joint pdf)* denoted by $f_{X,Y}(x, y)$. The pdf has the following properties:

1. $f_{X,Y}(x, y) \geq 0$ for every $(x, y) \in R^2$.
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$.
3. For any region A in the xy-plane $P((X, Y) \in A) = \int \int_A f_{X,Y}(x, y) dx dy$.

Marginal distributions. Let (X, Y) be a continuous/discrete random vector having a joint distribution with pdf/pmf $f(x, y)$. Then, the one-dimensional distributions of X and Y are called *marginal distributions*. We compute the marginal distributions as follows:

If (X, Y) is a discrete vector, then the distributions of X and Y are given by:

$$f_X(x) = \sum_{\text{all y}} P(X = x, Y = y) \text{ and } f_Y(y) = \sum_{\text{all x}} P(X = x, Y = y).$$

If (X, Y) is a continuous vector, then the distributions of X and Y are given by:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \text{ and } f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx.$$

Joint cdf of a vector (X, Y). The joint cumulative distribution function of X and Y (or *joint cdf*) is defined by

$$F_{X,Y}(u, v) = P(X \leq u, Y \leq v).$$

Theorem. Let $F_{X,Y}(u, v)$ be a joint cdf of the vector (X, Y). Then the joint pdf of (X, Y), $f_{X,Y}$, is given by second partial deriveative of the cdf. That is $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$, provided that $F_{X,Y}(x, y)$ has continuous second partial derivatives.

Multivariate random vectors. Please read on page 215.

INDEPENDENT RANDOM VARIABLES

Definition. Two random variables are called independent iff for every intervals A and B on the real line $P(X \in A \text{ and } Y \in B) = P(X \in A)P(Y \in B)$.

Theorem. The random variables X and Y are independent iff

$$f_{X,Y}(x, y) = f_X(x) f_Y(y),$$

where $f(x, y)$ is the joint pdf of (X, Y), and $f_X(x)$ and $f_Y(y)$ are the marginal densities of X and Y, respectively.

NOTE: Random variables X and Y are independent iff $F_{X,Y}(x, y) = F_X(x)F_Y(y)$, where $F(x, y)$ is the joint cdf of (X, Y), and $F_X(x)$ and $F_Y(y)$ are the marginal cdf's of the X and Y, respectively.

Independence of more than 2 r.v.s A set of n random variables $X_1, X_2, X_3, \ldots, X_n$ are independent iff their joint pdf is a product of the marginal pdfs, that is $f_{X_1,X_2,X_3,\ldots,X_n}(x_1, x_2, x_3, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$, where $f_{X_1,X_2,X_3,\ldots,X_n}(x_1, x_2, x_3, \ldots, x_n)$ is the joint pdf of the vector $(X_1, X_2, X_3, \ldots, X_n)$,
and $f_{X_1}(x_1)$, $f_{X_2}(x_2), \cdots, f_{X_n}(x_n)$ are the marginal pdf's of the variables $X_1, X_2, X_3, \ldots, X_n$.

Random Sample. A random sample of size n from distribution f is a set $X_1, X_2, X_3, \ldots, X_n$ of independent and identically distributed (iid), with distribution f, random variables.