4.5 Routh-Hurwitz Criteria

Important criteria that give necessary and sufficient conditions for all of the roots of the characteristic polynomial (with real coefficients) to lie in the left half of the complex plane are known as the *Routh-Hurwitz criteria*. The name refers to E. J. Routh and A. Hurwitz, who contributed to the formulation of these criteria. In 1875, Routh, a British mathematician, developed an algorithm to determine the number of roots that lie in the right half of the complex plane (Gantmacher, 1964). In 1895, Hurwitz, a German mathematician, verified the determinant criteria for roots to lie in the left half of the complex plane. According to Theorem 4.3, if the roots of the characteristic polynomial lie in the left half of the complex plane, then any solution to the linear, homogeneous differential equation converges to zero. The Routh-Hurwitz criteria for differential equations are analogous to the Jury conditions for difference equations. The Routh-Hurwitz criteria are used in Chapters 5 and 6 to determine local asymptotic stability of an equilibrium for nonlinear systems of differential equations. The Routh-Hurwitz criteria are stated in the next theorem.

Theorem 4.4 **(Routh-Hurwitz Criteria).** *Given the polynomial,*

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n,$$

where the coefficients a_i are real constants, $i = 1, \ldots, n$, define the n Hurwitz matrices using the coefficients a_i of the characteristic polynomial:

$$H_1 = (a_1), \quad H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \quad H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix},$$

and

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{pmatrix},$$

where $a_j = 0$ if $j > n$. All of the roots of the polynomial $P(\lambda)$ are negative or have negative real part iff the determinants of all Hurwitz matrices are positive:

$$\det H_j > 0, \quad j = 1, 2, \ldots, n.$$

When $n = 2$, the Routh-Hurwitz criteria simplify to $\det H_1 = a_1 > 0$ and

$$\det H_2 = \det \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} = a_1 a_2 > 0$$

or $a_1 > 0$ and $a_2 > 0$. For polynomials of degree $n = 2, 3, 4$ and 5, the Routh-Hurwitz criteria are summarized.

Routh-Hurwitz criteria for $n = 2, 3, 4,$ *and* 5.

$n = 2$: $a_1 > 0$ and $a_2 > 0$.

$n = 3$: $a_1 > 0$, $a_3 > 0$, and $a_1a_2 > a_3$.

$n = 4$: $a_1 > 0$, $a_3 > 0$, $a_4 > 0$, and $a_1a_2a_3 > a_3^2 + a_1^2a_4$.

$n = 5$: $a_i > 0$ $i = 1, 2, 3, 4, 5$, $a_1a_2a_3 > a_3^2 + a_1^2a_4$, and
$(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_5(a_1a_2 - a_3)^2 + a_1a_5^2$.

For a proof of the Routh-Hurwitz criteria, please see Gantmacher (1964). Theorem 4.4 is verified in the case $n = 2$.

Proof of Theorem 4.4 For $n = 2$, the Routh-Hurwitz criteria are just $a_1 > 0$ and $a_2 > 0$. The characteristic polynomial in the case $n = 2$ is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0.$$

The eigenvalues satisfy

$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}.$$

Suppose a_1 and a_2 are positive. It is easy to see that if the roots are real, they are both negative, and if they are complex conjugates, they have negative real part.

Next, to prove the converse, suppose the roots are either negative or have negative real part. Then it follows that $a_1 > 0$. If the roots are complex conjugates, $0 < a_1^2 < 4a_2$, which implies that a_2 is also positive. If the roots are real, then since both of the roots are negative it follows that $a_2 > 0$. □

Necessary but not sufficient conditions for the roots of the polynomial $P(\lambda)$ to lie in the left half of the complex plane are that the coefficients of $P(\lambda)$ be strictly positive. This result is stated in the next corollary.

Corollary 4.1 *Suppose the coefficients of the characteristic polynomial are real. If all of the roots of the characteristic polynomial*

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$$

are negative or have negative real part, then the coefficients $a_i > 0$ *for* $i = 1, 2, \ldots, n$.

Proof The corollary is a direct consequence of the Routh-Hurwitz criteria but can be verified separately. The characteristic equation can be factored into the form

$$(\lambda + r_1)\cdots(\lambda + r_{k_1})(\lambda^2 + 2c_1\lambda + c_1^2 + d_1^2)\cdots(\lambda^2 + 2c_{k_2}\lambda + c_{k_2}^2 + d_{k_2}^2) = 0,$$

where the real roots are $-r_i < 0$ for $i = 1, \ldots, k_1$ and the complex roots are $-c_j \pm d_ji$ for $j = 1, \ldots, k_2$ and $k_1 + 2k_2 = n$. If all of the roots are either negative or have negative real part, then $r_i > 0$ and $c_j > 0$ for all i and j. Thus, all the coefficients in the factored characteristic equation are