General Solutions to DE: Properties of a Set of Solutions

Each type of DE covered, has method of solving for general solution (see the section in the last column for details). A general solution is a way to write a solution to DE, which classifies all possible solutions; that is, describing the set of solution to a DE.

Initial Value Problems (IVP)

An initial value problems (IVP) is problem which is of this form: Find a function that is a solution to a differential equation, which also satisfies initial conditions. For example,

$$\text{(IVP)} \qquad \begin{gathered} n^{\text{th}} \text{ order DE} \\ y(t_0) = \alpha_0,\, y'(t_0) = \alpha_1,\, \ldots,\, y^{(n-1)}(t_0) = \alpha_{n-1}, \end{gathered}$$

Boundary Value Problems (BVP)

A boundary value problem is problem which is of this form: Find a function that is a solution to a differential equation, which also satisfies boundary conditions. For example,

$$\text{(BVP)} \qquad \begin{gathered} y'' + p(x)y' + q(x)y = g(x) \\ y(x_0) = \alpha_0 \quad \& \quad y(x_f) = \alpha_f \end{gathered}$$

where $x_0 < x_f$. Note: the boundary conditions can have condition(s) on the value of the derivative at either x_0 or x_f, as well.

Heat Equation and its BVP

The heat equation, a single type of partial differential equation (PDE), is mathematical model for heat conduction in a rod; many times, we are given information about boundary conditions. For example,

$$\begin{aligned} &\alpha^2 u_{xx}(x,t) = u_t(x,t) \qquad 0 < x < L, \quad t > 0 \\ &u(0,t) = g_0(t) \quad \& \quad u(L,t) = g_L(t) \\ &u(x,0) = f(x) \end{aligned}$$

About Properties of a Set of Solutions

We learn about properties of a set of solution to a differential equation. It is important to know the following:

1. Definition of *linearly dependence* and *independence*.

2. How to compute the *Wronskian*.

3. Theorems relating the Wronskian to (linearly) dependence and independence.

About General Solutions to DE

1. Each method to solve 1^{st}-order DE in the table requires good integration skills.

2. **Homogeneous with Constant Coef.** Use the *Characteristic Equation*; requires good factoring skill.

3. **Nonhomogeneous with Constant Coef.** Use the above bullet first to find the homogeneous solution. Then uses method of *undetermined coefficients* or *variation of parameters* which requires solving a system of equations or computing a Wronskian, respectively.

4. **Euler equation.** Use the *indicial equation*.

5. **Series solutions.** Find *recurrence relation* for the coefficients, so one can find the term of the power series.

About IVP

Most of the IVP we have had in this course are solved in two steps: first step, find a general solution to the DE; second step, use the initial conditions to find the exact solution which solves the IVP. We learned how to use Laplace Transform to solve IVP's.

It is important to remember the following:

1. The general solution (usually) has arbitrary constants c_i (for example, $y_{\text{gen}}(t) = c_1 e^{2t} + c_2 e^t$ has arbitrary constant c_1 and c_2). We use the initial conditions to find each c_i.

2. Solving an IVP with *Laplace Transform*, one has to use the table on page 319 of the textbook.

3. When $u_c(t)$ (step or *Heaviside function*) and/or $\delta(t-c)$ (*impulse* or *delta generalize-function*) are present in the DE, we have to use Laplace Transform to solve the IVP.

About BVP

Most of the BVP, we have had in this course are solved in two steps: first step, find a general solution to the DE; second step, use the boundary conditions to find all possible solution which solves the BVP. We look at a BVP with an DE with an arbitrary constant (for example, $y'' + \lambda y = 0$); in which, *eigenvalues* and *eigenfunctions* are defined.

About the Heat Equation

Solving the Heat equation uses *separation of variables*, then *eigenvalues* and *eigenfunctions*; when there is a BVP, we used *Fourier Series*.

Example

This is an example of a *series solution at a regular singular pt.*
Given

$$\begin{aligned}
&2x^2y'' + 3xy' + (2x^2-1)y = 0 \\
&y'' + \frac{3x}{2x^2}y' + \frac{(2x^2-1)}{2x^2}y = 0 && \text{mult. by } x^2 \\
&x^2y'' + x\left[x\frac{3x}{2x^2}\right]y' + \left[x^2\frac{(2x^2-1)}{2x^2}\right]y = 0 \\
&x^2y'' + x\left[xp(x)\right]y' + \left[x^2q(x)\right]y = 0,
\end{aligned}$$

where p and q are as follows:

$$\begin{aligned}
p(x) &= \frac{3}{2}\left(\frac{1}{x}\right) & q(x) &= 1 - \frac{1}{2x^2} \\
xp(x) &= \frac{3}{2} & x^2q(x) &= x^2 - \frac{1}{2}.
\end{aligned}$$

So we have

$$x^2y'' + x\left[\frac{3}{2}\right]y' + \left[x^2 - \frac{1}{2}\right]y = 0.$$

Assume that $y = \sum_{n=0}^{\infty} a_n x^{r+n}$ is a solution to the DE, where $a_0 \neq 0$. Thus

$$\begin{aligned}
&x^2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n-2} + \left[\frac{3}{2}\right]x\sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1} + \left[-\frac{1}{2} + x^2\right]\sum_{n=0}^{\infty}a_nx^{r+n} = 0 \\
&\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n} + \frac{3}{2}\sum_{n=0}^{\infty}(r+n)a_nx^{r+n} - \frac{1}{2}\sum_{n=0}^{\infty}a_nx^{r+n} + \sum_{n=0}^{\infty}a_nx^{r+n+2} = 0 \\
&\sum_{n=0}^{\infty}\left((r+n)(r+n-1) + \frac{3}{2}(r+n) - \frac{1}{2}\right)a_nx^{r+n} + \sum_{n=2}^{\infty}a_{n-2}x^{r+n} = 0
\end{aligned}$$

$$\left((r)(r-1)+\frac{3}{2}(r)-\frac{1}{2}\right)a_0x^r+$$
$$\left((r+1)(r)+\frac{3}{2}(r+1)-\frac{1}{2}\right)a_1x^{r+1}+$$
$$\sum_{n=2}^{\infty}\left((r+n)(r+n-1)+\frac{3}{2}(r+n)-\frac{1}{2}\right)a_nx^{r+n}+\sum_{n=2}^{\infty}a_{n-2}x^{r+n}=0$$
$$\left((r)(r-1)+\frac{3}{2}(r)-\frac{1}{2}\right)a_0x^r+$$
$$\left((r+1)(r)+\frac{3}{2}(r+1)-\frac{1}{2}\right)a_1x^{r+1}+$$
$$\sum_{n=2}^{\infty}\left[\left((r+n)(r+n-1)+\frac{3}{2}(r+n)-\frac{1}{2}\right)a_n+a_{n-2}\right]x^{r+n}=0.$$

For this sum to be 0 for all x in an interval shows that each coefficient of x^{r+n} in the sum has to be 0. Since we assumed that $a_0\neq 0$, we have that

$$(r)(r-1)+\frac{3}{2}(r)-\frac{1}{2}=\left(r-\frac{1}{2}\right)(r+1)=0,$$

So we have that $r=1/2$ or $r=-1$.

First Case: $r=1/2$.

$$\left((r+1)(r)+\frac{3}{2}(r+1)-\frac{1}{2}\right)a_1=\frac{5}{2}a_1=0,$$

which implies that $a_1=0$. Moreover,

$$\left((r+n)(r+n-1)+\frac{3}{2}(r+n)-\frac{1}{2}\right)a_n+a_{n-2}=0$$
$$n(n+\frac{3}{2})a_n=-a_{n-2}$$
$$(*)\qquad a_n=\frac{-2}{n(2n+3)}a_{n-2}.$$

Now we use $(*)$ to get a series

$$a_0y_{1/2}(x)=x^{1/2}\sum_{n=0}^{\infty}a_nx^n.$$

We see that $a_{2m+1}=0$, and we have

$$a_{2m}=\frac{-1}{m(4m+3)}a_{2(m-1)}=a_0\prod_{k=1}^{m}\frac{-1}{k(4k+3)}$$

Second Case: $r = -1$.

$$\left((r+1)(r) + \frac{3}{2}(r+1) - \frac{1}{2}\right) a_1 = -\frac{1}{2} a_1 = 0,$$

which implies that $a_1 = 0$. Furthermore,

$$\left((r+n)(r+n-1) + \frac{3}{2}(r+n) - \frac{1}{2}\right) a_n + a_{n-2} = 0$$

$$n(n - \frac{3}{2}) a_n = -a_{n-2}$$

$$(\dagger) \qquad a_n = \frac{-2}{n(2n-3)} a_{n-2}.$$

Now we use (†) to get a series

$$a_0 y_{-1}(x) = x^{-1} \sum_{n=0}^{\infty} a_n x^n.$$

We see that $a_{2m+1} = 0$, and we have

$$a_{2m} = \frac{-1}{m(4m-3)} a_{2(m-1)} = a_0 \prod_{k=1}^{m} \frac{-1}{k(4k-3)}$$