MATH 461/661 Homework #6 **Solutions**

In order to receive full credit, you must (1) present the solutions in the order of the assigned problems, (2) write or type legibly, and (3) justify your answers appropriately.

1. **3.5.12** $f_Y(y) \geq 0$ and $\int_1^\infty 1/y^2 dy = 1/y|_1^\infty = 1$, so $f_Y(y)$ is a pdf. However,

$$E(Y) = \int_1^\infty y f_Y(y) dy = \int_1^\infty \frac{1}{y} dy = \ln(\infty) - \ln(1) = \infty.$$

2. **3.5.14** X is binomial with $n = 15$ and $p = \int_{1/2}^1 3y^2 dy = 7/8$. Therefore

$$E(X) = np = \mathbf{105/8}.$$

3. **3.5.17** Since any pair is equally likely, we can just count how many pairs yield each possible X value. This yields $P(X = k) = \frac{(k-1)}{6}$, for $k \in \{2, 3, 4\}$ and thus

$$E(X) = 2(1/6) + 3(2/6) + 4(3/6) = \mathbf{10/3}$$

4. **3.5.25** X is hypergeometric with parameters r, w and n. Therefore, factoring factorials appropriately, we have that

$$E(X) = \sum_{k=0}^{r} \frac{\binom{r}{k}\binom{w}{n-k}}{\binom{r+w}{n}} k = \sum_{k=1}^{r} \frac{\binom{r}{k}\binom{w}{n-k}}{\binom{r+w}{n}} k = \frac{rn}{r+w} \sum_{k=1}^{r} \frac{\binom{r-1}{k-1}\binom{w}{n-k}}{\binom{r-1+w}{n-1}}$$

Using $R = r - 1$, $K = k - 1$, and $N = n - 1$ we can rewrite the sum

$$\sum_{k=1}^{r} \frac{\binom{r-1}{k-1}\binom{w}{n-k}}{\binom{r-1+w}{n-1}} = \sum_{K=0}^{R} \frac{\binom{R}{K}\binom{w}{N-K}}{\binom{R+w}{N}}$$

This, however, is just the sum of a hypergeometric distribution with parameters R, w, and N, which sums to 1. Thus,

$$E(X) = \frac{rn}{r+w}$$

5. **3.5.26** See also the example done in class when we covered properties of expectation.

$$\begin{aligned}
E(X) = \sum_{k=0}^{\infty} kp_k &= p_1 + p_2 + p_2 + \cdots + \overbrace{p_k + \cdots + p_k}^{k \text{ terms}} + \cdots \\
&= p_1 + p_2 + p_3 + p_4 + \cdots \\
&\qquad + p_2 + p_3 + p_4 + \cdots \\
&\qquad\qquad + p_3 + p_4 + \cdots \\
&\qquad\qquad\qquad + p_4 + \cdots \\
&= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \cdots \\
&= \sum_{k=1}^{\infty} P(X \geq k)
\end{aligned}$$

6. **3.5.31** In units of \$100K

$$
\begin{aligned}
E(Q(y)) = \int_0^\infty Q(y)f(y)dy &= \int_0^\infty 2(1-e^{-2y})\,6e^{-6y}dy \\
&= 12\int_0^\infty e^{-6y}dy - 12\int_0^\infty e^{-8y}dy \\
&= 2\int_0^\infty 6e^{-6y}dy - \frac{12}{8}\int_0^\infty 8e^{-8y}dy \\
&= 2 - \frac{12}{8} = \mathbf{\frac{1}{2}}
\end{aligned}
$$

Therefore their expected profit is \$50,000$^{.00}$.

7. **3.5.35** Since $Y^2 = a^2 + a^2 = 2a^2$, area $A(Y) = a^2/2 = Y^2/4$. The pdf of Y is $1/4$ over [6,10]. Therefore

$$
E(A(Y)) = \int_6^{10} \frac{y^2}{4}\cdot\frac{1}{4}\,dy = \frac{1}{48}\int_6^{10} 3y^2\,dy = \frac{1}{48}(10^3 - 6^3) = \mathbf{\frac{49}{3}} = \mathbf{16.\bar{3}}
$$

8. **3.6.4** $Var(X) = E((X-\frac{1}{2})^2) = \int_0^1 (x-\frac{1}{2})^2\,dx = \int_0^1 x^2 - x + \frac{1}{4}\,dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \mathbf{\frac{1}{12}}$

9. **3.6.11** $Var(Y) = E(Y^2) - E(Y)^2$, and we know $E(Y) = \frac{1}{\lambda}$. Using integration by parts,

$$
E(Y^2) = \int_0^\infty y^2\lambda e^{-\lambda y}\,dy = \underbrace{\left[-y^2e^{-\lambda y}\right]_0^\infty}_{=0} - \underbrace{\left[-\frac{y}{\lambda}e^{-\lambda y}\right]_0^\infty}_{=0} + \frac{2}{\lambda^2} = \frac{2}{\lambda^2}
$$

Therefore

$$
Var(Y) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}
$$

10. **3.6.15** Since the standard deviation is the square root of the variance, then letting σ_C and σ_F denote the standard deviation in Celsius and Fahrenheit, respectively, Theorem 3.6.2 gives that

$$
\sigma_C = \sqrt{Var\left(\frac{5}{9}(Y-32)\right)} = \frac{5}{9}\sqrt{Var(Y)} = \frac{5}{9}\sigma_F
$$

and thus

$$
\sigma_C = \frac{5}{9}15.7 = \mathbf{\frac{157}{18}} = \mathbf{8.7\bar{2}\,^\circ C}
$$