# Example Probability Distributions

| **Name (Discrete)** | **S** | **$\Theta$** | **PMF** | **CDF** | **E(X)** | **Var(X)** | **$\mathbf{M_X(t)}$** |
|---|---|---|---|---|---|---|---|
| Bernoulli | $x \in \{0,1\}$ | $p$ | $p^x(1-p)^{1-x}$ | - | $p$ | $p(1-p)$ | $1-p+p\,e^t$ |
| Binomial | $\{0,...,n\}$ | $n,\, p$ | $\binom{n}{x}p^x(1-p)^{n-x}$ | - | $np$ | $np(1-p)$ | $(1-p+p\,e^t)^n$ |
| Multinomial | $\sum X_i = n$ | $n, \sum p_i = 1$ | $\frac{n!}{x_1!\cdots x_k!}p_1^{x_1}\cdots p_k^{x_k}$ | - | $E(X_i) = np_i$ | $V(X_i) = np_i(1-p_i)$ | $\left(\sum_{i=1}^k p_i e^{t_i}\right)^n$ |
| Hypergeometric | $\{0,...,n\}$ | $n,\, w,\, N$ | $\binom{w}{x}\binom{N-w}{n-x}/\binom{N}{n}$ | - | $n\,(w/N)$ | $n(w/N)(1-w/N)\frac{N-n}{N-1}$ | - |
| Gen. Hypergeometric | $\{0,...,n\}^k$ | $N = \sum n_i,\, n$ | $\binom{n_1}{x_1}\cdots\binom{n_k}{x_k}/\binom{N}{n}$ | - | Marginals are Hypergeometric | | - |
| Geometric (# failures) | $\{0,1,...\}$ | $p$ | $(1-p)^x p$ | $1-(1-p)^{k+1}$ | $(1-p)/p$ | $(1-p)/p^2$ | $\frac{p}{1-(1-p)\exp(t)}$ |
| Negative Binomial | $\{0,1,...\}$ | $n,\, p$ | $\binom{x+n-1}{x}p^n(1-p)^x$ | - | $n(1-p)/p$ | $n(1-p)/p^2$ | $\left(\frac{p}{1-(1-p)\exp(t)}\right)^n$ |
| Negative Binomial | $\{0,1,...\}$ | $n,\, \mu$ | $\frac{\Gamma(n+x)}{\Gamma(n)x!}(\frac{n}{n+\mu})^n(\frac{\mu}{n+\mu})^x$ | - | $\mu$ | $\mu+\mu^2/n$ | $\left(\frac{n}{n+\mu-\mu\,e^t}\right)^n$ |
| Geometric (# trials) | $\{1,2,...\}$ | $p$ | $(1-p)^{x-1}p$ | $1-(1-p)^k$ | $1/p$ | $(1-p)/p^2$ | $\frac{p\exp(t)}{1-(1-p)\exp(t)}, \quad t < \ln\left(\frac{1}{1-p}\right)$ |
| Negative Binomial | $\{n,...\}$ | $n,\, p$ | $\binom{x-1}{n-1}p^n(1-p)^{x-n}$ | - | $n/p$ | $n(1-p)/p^2$ | $(\frac{p\exp(t)}{1-(1-p)\exp(t)})^n,\ t < \ln\left(\frac{1}{1-p}\right)$ |
| Poisson($rT$) | $\{0,1,...\}$ | $r,\, T$ | $e^{-rT}\frac{(rT)^x}{x!}$ | - | $rT$ | $rT$ | $\exp(rT(e^t-1))$ |
| Poisson($\lambda$) | $\{0,1,...\}$ | $\lambda$ | $e^{-\lambda}\frac{\lambda^x}{x!}$ | - | $\lambda$ | $\lambda$ | $\exp(\lambda(e^t-1))$ |

| **Name (Continuous)** | **S** | **$\Theta$** | **PDF** | **CDF** | **E(X)** | **Var(X)** | **$\mathbf{M_X(t)}$** |
|---|---|---|---|---|---|---|---|
| Uniform | $[a,b]$ | $a,\, b$ | $\frac{1}{b-a}$ | $\frac{x-a}{b-a}$ | $\frac{a+b}{2}$ | $\frac{1}{12}(b-a)^2$ | $\frac{e^{tb}-e^{ta}}{t(b-a)}$ |
| Exponential (rate $r$) | $[0,\infty)$ | $r$ | $r\,e^{-rx}$ | $1-e^{-rx}$ | $\frac{1}{r}$ | $\frac{1}{r^2}$ | $(1-t/r)^{-1}$ |
| Exponential (mean $\theta$) | $[0,\infty)$ | $\theta$ | $1/\theta\,e^{-x/\theta}$ | $1-e^{-x/\theta}$ | $\theta$ | $\theta^2$ | $(1-\theta t)^{-1}$ |
| Gamma(shape $k$, rate $r$) | $[0,\infty)$ | $k,\, r$ | $\frac{r^k}{\Gamma(k)}x^{k-1}e^{-rx}$ | - | $\frac{k}{r}$ | $\frac{k}{r^2}$ | $(1-t/r)^{-k}$ |
| Gamma(shape $k$, scale $\theta$) | $[0,\infty)$ | $k,\, \theta$ | $\frac{\theta^{-k}}{\Gamma(k)}x^{k-1}e^{-x/\theta}$ | - | $k\theta$ | $k\theta^2$ | $(1-\theta t)^{-k}$ |
| Gamma(shape $\alpha$, rate $\beta$) | $[0,\infty)$ | $\alpha,\, \beta$ | $\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ | - | $\frac{\alpha}{\beta}$ | $\frac{\alpha}{\beta^2}$ | $(1-t/\beta)^{-\alpha}$ |
| Normal | $\mathbb{R}$ | $\mu,\, \sigma$ | $\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ | - | $\mu$ | $\sigma^2$ | $\exp(\mu t+\frac{1}{2}\sigma^2t^2)$ |
| Beta | $[0,1]$ | $a,\, b$ | $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$ | - | $\frac{a}{a+b}$ | $\frac{ab}{(a+b)^2(a+b+1)}$ | - |
| Pareto | $[x_m,\infty)$ | $x_m,\, \alpha$ | $\alpha x_m^\alpha/x^{\alpha+1}$ | $1-\left(\frac{x_m}{x}\right)^\alpha$ | $\frac{\alpha x_m}{\alpha-1}$; ($\infty$ if $\alpha \le 1$) | $\frac{\alpha\,x_m^2}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$ | - |

**Note:** Quantities not shown include: *median*, *mode*, *quantile function*, *skewness*, *kurtosis*, *entropy*, *characteristic function* (Fourier Transformed PDF), *conjugate prior*.

# Generalized Applications

| Name (Discrete) | Application |
|---|---|
| Bernoulli | Binary (i.e., 0 or 1) outcome, with probability $p$ of focal outcome (aka 'success'). That is, $\mathcal{P}(X = 1) = p$. |
| Binomial | The distribution of # successes in $n$ Bernoulli trials. 0 to $n$ successes possible. Ex: Flipping a coin $n$ times, counting "heads". |
| Multinomial | Like binomial, but for $k$ outcome types, not just 2 outcomes (i.e, 0 or 1). Ex: Distribution of outcomes from rolling a $k$-sided dice $n$ times. |
| Hypergeometric | Ex: Distribution of # black balls obtained by drawing (*without* replacement) $n$ balls from an urn containing $K$ black and $N - K$ white balls. |
| Gen. Hypergeometric | Like Hypergeometric (above), but with multiple ($k > 2$) colors of balls. (Hypergeometric is just the $k = 2$ case) |
| Geometric (# failures) | Ex: Number of failures before a success, where probability of success at each trial is $p$. |
| Negative Binomial | Like Geometric (above), but the number of failures (not number of trials!) before the $n^{\text{th}}$ success. ($n = 1$ gives the above distribution.) |
| Negative Binomial | Same as above (i.e., count of # failures before $n^{\text{th}}$ success) but parameterized in terms of the mean $\mu$ instead of success probability $p$. |
| Geometric (# trials) | Variant of Geometric (see above), but for counting the number of *total trials* (including the successful trial), not just failures. |
| Negative Binomial | Variant of Negative Binomial above, but counts *total trials* (including successes) taken to reach the $n^{\text{th}}$ success (alt. parameterization not shown). |
| Poisson($rT$) | Count of events in interval $[0, T]$ where the inter-event intervals are exponentially distributed with rate $r$ (i.e., exponential with mean $1/r$.) |
| Poisson($\lambda$) | Same as above, but parameterized in terms of the expected (mean) number of events ($\lambda$). |

| Name (Continuous) | Application |
|---|---|
| Uniform | All outcomes are equally likely. |
| Exponential (rate $r$) | Continuous version Geometric Distribution: Time duration until an event. Over small time interval $\Delta t$, the probability of the event is $p \approx r\,\Delta t$. |
| Exponential (mean $\theta$) | Same as above, but parameterized in terms of the mean duration time $\theta = 1/r$. |
| Gamma(shape $k$, rate $r$) | Continuous version of Negative Binomial: Time until $k^{\text{th}}$ event, where probability of event in $\Delta t$ is $\approx r\,\Delta t$. **Alt:** Sum of $k$ exponentials (rate $r$). |
| Gamma(shape $k$, scale $\theta$) | Same as above, but follows the alternate parameterization of the Exponential using it's mean, not rate. **Alt:** Sum of $k$ exponentials (mean $\theta$). |
| Gamma(shape $\alpha$, rate $\beta$) | Same as above. A common alternative parameterization that just uses different notation ($\alpha$ and $\beta$) for the shape ($k$) and rate ($r$), respectively. |
| Normal | The **Central Limit Theorem** says sums of iid r.v.s are approximately Normal. Ex: Many data that reflect multiple sources of "randomness" |
| Beta | In Bayesian statistics, its the *conjugate prior distribution* for estimates of $p$ in Bernoulli, binomial, geometric, and negative binomial distributions. |
| Pareto | A Power Law distribution used to model heavy-tailed data. If $X$ is exponential (rate $\alpha$), then $Y = x_m\, e^X$ is Pareto. |

Updated: 3 December 2015 by Paul J. Hurtado